Collapse mechanism maps for the hollow pyramidal core of a sandwich panel under transverse shear

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The finite element method has been used to develop collapse mechanism maps for the shear response of sandwich panels with a stainless steel core comprising hollow struts. The core topology comprises either vertical tubes or inclined tubes in a pyramidal arrangement. The dependence of the elastic and plastic buckling modes upon core geometry is determined, and optimal geometric designs are obtained as a function of core density. For the hollow pyramidal core, strength depends primarily upon the relative density of the core with a weak dependence upon tube slenderness. At relative density above 3%, the tubes of the pyramidal core buckle plastically and the peak shear strength scales linearly with density. In contrast, at relative density below 3%, the tubes do not buckle and a stable shear response is observed. The predictions of the current study are in excellent agreement with previous measurements on the shear strength of the hollow pyramidal core, and suggest that this core topology is attractive from the perspectives of both core strength and energy absorption.

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1. Introduction

Metallic sandwich plates bring structural benefit over their monolithic counterparts due to increased structural stiffness and strength, and the potential for multifunctional application. For example, sandwich plates have potential application as structural armour in land, sea and air vehicles for providing structural stiffness, and resistance to crash, blast and ballistic attack. Typically, sandwich panels are loaded by spatially varying transverse loads, and consequently the core of the panel must possess adequate compressive strength and longitudinal shear strength. The properties of the core are sensitive to both material choice and topology, and in the current study we shall explore the use of hollow tubes made from stainless steel as candidate core material. Stainless steels have both high ductility and corrosion resistance, and a wide range of yield strengths (200–1000 MPa), depending upon the alloy and heat treatment.

Lattices are mechanically competitive alternatives to prismatic (corrugated) and honeycomb structures when configured as the core of a sandwich panel. There has been significant recent activity in the invention, manufacture and testing of new core topologies, see for example the review by Fleck et al. (2010) and Wadley et al. (2003a). Lattice sandwich structures are of particular interest because of their fully open interior structure which facilitates multifunctional applications (Evans et al., 1998a,b; Wadley, 2006). For example, lattice core sandwich panels are capable of supporting significant structural loads while also facilitating cross flow heat exchange (Kim et al., 2005). Sandwich panels with lattice cores made from hollow stainless steel tubes are well-suited for structural heat exchangers: stainless steel combines structural and thermal performance along with high corrosion resistance. The lattice topology may also alleviate some of the delamination and corrosion concerns associated with the use of traditional closed cell honeycomb sandwich panels (Blitzer, 1997).

Early experimental studies on lattice-cored sandwich panels were limited to the manufacturing route of investment casting, and this restricted the material choice to high fluidity casting alloys such as high Si-content aluminium alloys (Deshpande and Fleck, 2001; Sugimura, 2004; Zhou et al., 2004) and to non-structural alloys such as the casting brasses (Chiras et al. (2002), Wallach and Gibson, 2001). However, the tortuosity of the lattices and ensuing casting porosity made it difficult to fabricate high quality structures at low relative densities (2–10%) identified as optimal for sandwich panel constructions (Chiras et al., 2002). In these early investigations, premature failure occurred from casting defects. The resulting tensile ductility was sufficiently low (a few percent) that shear loading of sandwich panels gave rise to brittle failure of the tensile struts rather than to elastic or plastic buckling of the compressive struts (Sugimura, 2004).

Efforts to exploit the inherent ductility and toughness of many wrought engineering alloys led to the development of alternative...
lattice fabrication approaches based upon perforated metal sheet folding (Wadley et al., 2003a). These folded truss structures can be bonded to each other or to face-sheets by conventional joining techniques such as brazing, transient liquid phase (TLP) bonding or welding techniques to form all metallic lattice truss sandwich panels. Panels fabricated from austenitic stainless steels with tetrahedral and pyramidal lattice truss topologies have been made by node row folding of a patterned sheet to form the core and TLP bonding to facesheets, see for example Lim and Kang (2006), Rathbun et al. (2004), McShane et al. (2006), Zok et al. (2004). Because of the high temperatures normally encountered with TLP bonding, this process results in sandwich panels which remain in a low strength, annealed condition. Consequently, they collapse by plastic buckling under compression or shear. While these structures appear much more robust than their investment cast counterparts, the reduced strength of their annealed microstructure can limit their potential uses for some structural applications. This has been addressed recently by the use of heat treatable aluminium alloys such as 6061-T6, see Kooistra et al. (2008).

1.1. The shear response of sandwich cores

Most studies on lattice-cored sandwich panels have concentrated upon the out-of-plane compressive response, which is important for understanding blast resistance and the indentation behaviour of sandwich panels. However, the shear response of the panel is of equal importance, as the bending moment distribution carried by a panel necessarily gives rise to transverse shear loading and to the possibility of collapse of the core in shear. Indeed, core collapse in shear can dominate the competing failure modes in sandwich beams and plates, see for example Ashby et al. (2000) and Deshpande and Fleck (2001).

1.1.1. Prismatic sandwich cores

Consider first the collapse of sandwich cores with a prismatic, 2D morphology. Honeycombs usually comprise hexagonal or square cells, with the prismatic direction normal to the face of the sandwich panel.

Hexagonal honeycombs are routinely employed as the cores for lightweight sandwich panels and as energy absorbers; they are typically manufactured from aluminium alloys and have a relative density \( \rho \) (ratio of the density of the honeycomb treated as a homogeneous continuum to the density of the solid) of less than 3%: experiments and simple analyses have shown that their out-of-plane elastic properties scale linearly with their relative density \( \rho \) (Kelsey et al., 1958, Zhang and Ashby, 1992). In out-of-plane crushing, these honeycombs exhibit a stress peak followed by large stress oscillations associated with the formation of a succession of plastic folds in each cell. Similarly, the out-of-plane peak shear strength is governed by cell wall buckling as discussed by Werren and Norris (1950) and Zhang and Ashby (1992). Once the wrinkles have formed, the shear stress drops and subsequently remains approximately constant until failure occurs by the fracture of the cell walls, see for example Mohr and Doyoyo (2004). Most experimental studies are restricted to relative densities \( \rho < 0.08 \) as debonding of the honeycombs from the face-sheets has been observed at higher relative densities (Werren and Norris, 1950, Zhang and Ashby, 1992).

Square honeycomb cores having a high relative density \( \rho > 0.05 \) are preferable to hexagonal honeycombs for high severity loadings such as blasts and shocks because of their high out-of-plane crushing resistance, shear resistance and high in-plane stretching strength, Fleck and Deshpande (2004) and Xue and Hutchinson (2004). Enhancements in the performance of square honeycombs are expected when constructed from solids of high strain hardening, such as stainless steels. An experimental investigation into the out-of-plane compressive response of stainless square honeycombs by Côté et al. (2004) over a relative density range \( 0.03 < \rho < 0.2 \) confirmed that the honeycombs exploit the strain hardening behaviour of the stainless steel with the peak compressive strength set by the axial torsional plastic buckling of the square honeycomb cells. In fact, no progressive folding of the cell walls was observed by Côté et al. (2004). This difference in compressive response between the aluminium and stainless steel honeycombs is attributed to differences in the strain hardening response of the parent materials. Côté et al. (2006b) subsequently determined the shear response of metallic square honeycomb as a function of relative density \( \rho \) and of the direction of shearing relative to the cell walls. They found that the square honeycomb topology has a high shear stiffness and a high shear strength despite the occurrence of plastic wrinkling in the cell walls. The collapse mode of plastic wrinkling in the cell walls gave a graceful collapse response with no peak load, and no progressive folding of the cell walls, at \( \rho > 3\% \).

A limited literature exists on the shear collapse response of the corrugated and diamond core lattices: for these topologies, the prismatic direction of the core lies within the plane of the facesheets. Côté et al. (2006a) studied the plastic collapse response of corrugated and diamond core topologies in type 304 stainless steel, under shear and compressive loading. They found that the longitudinal shear strength (shear direction aligned with the prismatic direction) was significantly higher than the transverse strength, and this was attributed to the differences in buckling modes.

Akisanya and Fleck (2006) have explored the shear response of metallic conical frusta subjected to shear loading. They noted that the energy absorption under shear is limited by the initiation of sheet-metal necking of the frustum wall. Their results are directly useful for predicting the collapse response of an egg-box core to shear loading.

1.1.2. 3D lattice cores

Second, consider the collapse of sandwich cores with a 3D lattice morphology. Deshpande and Fleck (2001) and Rathbun et al. (2004) have measured the structural response of metallic sandwich beams with a tetrahedral core, made from a cast aluminium alloy and 304 stainless steel, respectively. Likewise, Kooistra et al. (2007) conducted experiments on tetrahedral lattice core sandwich panels made from 6061 aluminium alloy, while Lim and Kang (2006) and Hyun et al. (2009) have explored the plastic collapse response of Kagome trusses in 304 stainless steel, manufactured by a wire-weaving technique. In all cases, plastic buckling of the truss members dictated the shear strength of the tetrahedral core. Recently, Biagi and Bart-Smith (2007) have considered imperfections in the form of random debonded nodes of a pyramidal core made from 304 stainless steel. This flaw is used to mimic errors in manufacture by brazing of nodes to the face sheets. They found that the shear strength drops in proportion to the number of debonded nodes: a net section strength criterion is observed, implying high damage tolerance.

1.2. The hollow pyramidal core

Queheillalt and Wadley (2005a,b) have recently emphasized the utility of the hollow pyramidal core made from circular cylinders, as the individual core struts possess enhanced buckling strength over their solid counterparts. Additionally, the post-buckling collapse response can be tuned to be benign, with no sudden drop in load, upon suitable choice of tube geometry to exploit the shell-effect. The pyramidal core comprises four inclined tubes, with unit cell as shown in Fig. 1a.

Subsequently, Queheillalt and Wadley (2011) have measured the compressive and shear strength of a sandwich panel containing...
a hollow pyramidal core made from type 304 stainless steel. These cores collapsed by plastic buckling in a number of different modes, dependent upon geometry and loading direction. Queheillalt and Wadley (2011) compared the compressive and shear strengths with previous results they obtained for the pyramidal core made from solid struts, Queheillalt and Wadley (2005a,b): for relative densities in the range 1%–10%, the hollow struts buckled plastically whereas the solid struts buckled elastically, and consequently the hollow struts had higher strengths (by a factor of about three). However, Queheillalt and Wadley (2011) tested only a limited number of geometries and it is unclear whether they determined the full range of collapse modes, and whether their geometries gave the optimal strength for a given relative density. The recent numerical study of Pingle et al. (2010) revealed a rich set of competing modes in compression for the 304 stainless steel. Also, they found that the compressive strength is primarily dependent upon the relative density of the hollow pyramidal core, with additional geometric variables (such as the wall thickness to diameter ratio) playing a secondary role. In this sense, the geometries considered by Queheillalt and Wadley (2011) can be considered to be close to optimal.

It is clear from the above assessment of the existing literature on the collapse of lattice-cored sandwich panels that studies have focussed on type 304 stainless steel as the parent material. There are several reasons for this:

(i) this austenitic stainless steel has high corrosion resistance making it attractive for marine applications such as ship hulls;
(ii) it has high formability, and is thereby amenable to folding and stretching operations required for truss manufacture (see Wadley et al., 2003a);
(iii) it can be brazed and welded to solid or lattice face-sheets;
(iv) slender trusses made from 304 stainless steel are resistant to plastic buckling due to its high strain hardening capacity.

The objective of the present study is to explore numerically the shear response of sandwich panels comprising a hollow pyramidal core or a 'tubular core' made from stainless steel; the tubular core is an array of identical hollow circular cylinders such that the axis of each cylinder is aligned with the face sheet normal. The shear strength of the hollow core depends upon the length \(l\), thickness \(t\) and outer diameter \(d\) of each tube, as defined in Fig. 1a. The tubes offer enhanced resistance to elastic and plastic buckling due to the increased radius of gyration compared to the solid counterparts. The face-sheets are taken to be rigid and thereby prevent any interaction of response from one pyramidal cell (or vertical tube) to the next. Consequently, the unit cell approach suffices, and the collapse modes of a tubular core in shear are the same as those for an isolated built-in tubular column under end shear. A number of collapse modes have been identified in the literature both in the elastic and plastic buckling regimes but there has been no systematic assembly of the overall buckling map for shear loading. We now briefly review the observed buckling modes of an isolated tube under end-shear.

1.3. A brief summary of the collapse behaviour of ductile circular tubes under pure transverse shear

The buckling of cylinders under transverse shear loads has received less attention than that of axial compression. In order to specify the problem, consider a cylindrical vertical tube of length...
outer diameter \( d \) and thickness \( t \), built-in at each end: the ends are subjected to a relative shear displacement, float freely in the axial direction (Fig. 2a) and are constrained against rotation. The transverse shear force gives rise to a bending moment on the cross-section that varies linearly along the length of the tube, and is a maximum at the ends.

A number of collapse modes have been identified from the literature, as follows. Thin-walled cylinders of low \( t/d \) buckle elastically in two modes: at large \( l/d \) the bending moment dominates and a bending instability occurs (Brazier (1926)), whereas at small \( l/d \) elastic buckling dominates, as identified by Lundquist (1935). In contrast, cylinders of moderate \( t/d \) buckle plastically in three modes:

(i) at large \( l/d \) a bending instability occurs by a combination of plastic Bending instability (see Gellin (1980)) and short wavelength rippling as identified by Kyriakides and Ju (1992a,b), Corona et al. (2006), Liman et al. (2010). Notwithstanding the complex interaction of the two modes, the peak bending moment is adequately approximated by the Brazier plastic moment as given by Gellin (1980) for a Ramberg–Osgood solid.

(ii) at intermediate \( l/d \) plastic shear buckling occurs, as first investigated by Galletly and Blachut (1985) and as reviewed by Teng (1996). The role of imperfections upon the collapse load has been extensively explored in this regime and found to be mild, see Murakami et al. (1993) and Teng (1996).

(iii) at small \( l/d \) a plastic wrinkling mode occurs, as observed by Coté et al. (2006b). For the case of 304 stainless steel, as considered by Coté et al. (2006b), the wrinkling mode is sufficiently benign to not induce a peak load.

1.4. Scope of present study

The present study focuses on the generation of collapse mechanism maps for two core topologies of sandwich panel: the pyramidal core made from inclined tubes, and the ‘tubular core’ made from vertical tubes that straddle the face sheets. The relative strength and energy absorption capacity of competing buckling modes are analysed as a function of tube geometry. The shear strengths and buckling modes of the hollow pyramidal core are then compared to the measurements of Queheillalt and Wadley (2011). Our study builds on the recent analysis by Pingle et al. (2010) on the compressive response of the hollow pyramidal core. In that study a number of elasto-plastic buckling modes were identified as a function of tube geometry. We shall contrast these buckling modes under remote compression with those observed here for remote shear.

The study begins with an investigation of the buckling modes of vertical tubes under transverse shear loading. Five regimes of buckling are determined and indicated on a collapse classification map. For the case of thin-walled tubes (\( t/d < 0.03 \)) analytical formulae are used to determine the boundaries between the buckling regimes.

The geometry of the hollow pyramidal core and boundary conditions for shear loading are described in Section 3. The FE modelling procedure to analyze the collapse modes of the pyramidal core is explained in Section 4, and a collapse mechanism chart is constructed. Contours of core relative density \( \rho \) and peak shear strength are then added to the collapse mechanism map, and the influence of the loading direction relative to the orientation of the pyramidal core is quantified. Finally, the fidelity of the FE simulations is gauged by comparing FE predictions with the recent experimental results of Queheillalt and Wadley (2011). A comparison of the shear strength is made for competing core topologies and the energy absorption capacity of the hollow pyramidal core is contrasted with that of metallic foams.

2. The collapse of vertical tubes under transverse shear loading

2.1. Vertical rod under transverse shear

Prior to giving a full numerical analysis of the collapse response of hollow tubes under end transverse shear, it is instructive to consider the reference problem of a vertical bar under shear. We consider the simplest case of a bar of solid section made from a rigid, ideally plastic solid of yield strength \( \sigma_Y \) and list the analytical solutions for the two extremes of slenderness ratio: the slender limit where beam theory applies and the stocky limit where the material elements within the strut are subjected to simple shear.

Consider a vertical circular rod of diameter \( d \) and length \( l \) with the ends subjected to a relative shear displacement \( u \) and zero relative rotation; see Fig. 3. Define the nominal wall stress \( \tau_w \) in terms of the transverse shear force \( F_t \) on the section and the initial cross-sectional area \( A_o \) such that \( \tau_w = F_t/A_o \). The nominal shear strain \( \gamma_n \) is defined as the end displacement \( u \) divided by the initial length \( l \). Consider the following two cases.

(i) The stubby rod

A stubby rod experiences uniform shearing at \( \tau_w = \sigma_Y/\sqrt{3} \), according to the usual von-Mises yield criterion. The nominal shear stress versus strain response (and deformed shape) is shown in Fig. 3 for such a stubby rod, with \( l/d = 0.10 \). We note in passing that this collapse state also prevails for the stubby tube, with the wall shear stress again given by \( \tau_w = \sigma_Y/\sqrt{3} \).

(ii) The slender rod

In contrast, a slender solid rod plastically collapses by the rotation of plastic hinge at each end, with the plastic collapse moment \( M_p \) given by

\[
M_p = \frac{1}{6} d^3 \sigma_Y \tag{2.1}
\]

A relative shear displacement \( u \) of the ends of the bar causes the vertical separation of the end faces to reduce from the value \( l \) to a height
Moment equilibrium dictates that the transverse shear force $F_s$ is related to the end moment $M_p$ according to

$$F_s = 2M_p/h$$  \hspace{1cm} (2.2)$$

and, upon making use of $\gamma_n = u/l$ along with (2.2) and (3) we obtain

$$\frac{\tau_w}{\sigma_Y} = \frac{4d}{3\pi l} (1 - \gamma_n^2)^{-1/2}$$  \hspace{1cm} (4)$$

This characteristic is sketched in Fig. 3 for the choice $l/d = 15$.

Note that the axial force $T$ in the bar is given by

$$T = \frac{u}{l} F_s$$  \hspace{1cm} (5)$$

and the bar yield axially when $T$ attains the value of

$$T = \frac{\pi d^2}{4} \sigma_Y$$  \hspace{1cm} (6)$$

Upon combining (4)–(6), we deduce that axial yield occurs when $\gamma_n$ satisfies the condition

$$F_s = \frac{T}{\gamma_n} = \frac{1}{3} \frac{d^3 \sigma_Y}{l} (1 - \gamma_n^2)^{-1/2}$$  \hspace{1cm} (7)$$

or

$$\gamma_n (1 - \gamma_n^2)^{-1/2} = \frac{3\pi l}{4d}$$  \hspace{1cm} (8)$$

Now for slender bars, such that $d/l$ is small, this implicit relation gives the asymptotic result

$$\gamma_n \approx 1 - \frac{8}{9\pi^2} \left( \frac{d}{l} \right)^2$$  \hspace{1cm} (9)$$

The corresponding value of shear stress is

$$\frac{\tau_w}{\sigma_Y} = 1$$  \hspace{1cm} (10)$$

We conclude that axial yielding only occurs after the slender strut has fully rotated from the vertical to the horizontal position. It is also evident that finite rotation of the bar leads to a stable macroscopic response for the ideally plastic solid, see Fig. 3.

The above analysis can be repeated for a slender vertical tube of outer diameter $d$ and wall thickness $t$. For example, the plastic moment in (2.1) is now given by

$$M_p = \frac{1}{6} d^3 - (d - t)^3 \sigma_Y$$  \hspace{1cm} (11)$$

and the wall stress now reads

$$\frac{\tau_w}{\sigma_Y} = \frac{4F_s}{\pi [d^2 - (d - t)^2]}$$  \hspace{1cm} (12)$$

Upon substituting (2.1)–(3) and (11) into (12) we obtain

$$\frac{\tau_w}{\sigma_Y} = \frac{4}{3\pi} \left[ \frac{d^3 - (d - t)^3}{d^2 - (d - t)^2} \right] \left(1 - \gamma_n^2\right)^{-1/2}$$  \hspace{1cm} (13)$$

It is emphasized that this strength-of-materials approach should be viewed as an upper bound on the plastic collapse response as it neglects the possibility at local instabilities (such as ovalisation-softening and local rippling).

2.2. Finite element modelling

We now turn our attention to the transverse shear of a vertical tube (Fig. 2a) made from AISI 304 annealed stainless steel, making...
use of the true stress versus logarithmic strain response as measured by Queheillalt and Wadley (2011), as reproduced here in Fig. 2b. This response is characteristic of a conventional dislocation-hardening solid, with no upturns that would indicate phase transformation to martensite, see for example De et al. (2006).

Thus, conventional plasticity theory is used to model the multiaxial response of 304 stainless steel, as used by Rathbun et al. (2004) and Biagi and Bart-Smith (2007). Nonlinear finite element simulations have been performed using the implicit version of commercial finite element software, ABAQUS/Standard (version 6.6). The tubes are meshed with eight noded hexahedral linear element (C3D8R in ABAQUS notation) employing reduced integration. A mesh convergence study shows that average element size of $t/8$ gives accurate results. The self-contact option of ABAQUS is employed to prevent self-penetration of elements. The simulations have been performed in displacement-control, using the large deformation capability (NLGEOM) to capture the post-buckling response. (When snap-back occurs in elastic shear buckling, the Rik’s algorithm is employed to obtain the equilibrium path beyond peak load). The tube wall material is treated as an elastic–plastic solid satisfying J2 flow theory, and hardening characteristic as shown in Fig. 2b. Our study is limited to the case of perfect tubes without an added imperfection as our intent is to scope out the collapse mechanism map over a very wide range of geometries rather than to investigate a particular buckling mode in great detail. Further, load introduction is via end-clamped grips and these act to introduce their own imperfection.

2.3. Regimes of collapse of vertical tubes under transverse shear

A large number of FE simulations have been performed (160) in order to determine the sensitivity of collapse response to geometry. Six discrete deformation modes have been identified by visual inspection of the deformed meshes deep in the plastic range, and the regime of dominance of each mode is shown in Fig. 4. This approach has been used with success in previous experimental and theoretical studies on the compression of tubes (Andrews et al., 1983; Guillow et al., 2001; Pingle et al., 2010). The FE simulations for elastic buckling were in good agreement with analytical estimates taken from the literature and these are summarized in Appendix A. Two modes of elastic buckling are identified at small $t/d$ (less than about 0.001). An elastic bending instability occurs at large $l/d$, whereas elastic shear buckling dominates at small $l/d$. Both modes are unstable and give rise to a peak load. The current study focuses upon the practical regime of tubular cores with $t/d > 0.001$, and such tubes yield before they buckle elastoplastically. Post yield, they may or may not buckle plastically, and the precise details depends upon the values of $(t/d,l/d)$ as shown in Fig. 4 and as discussed below.

The nominal shear stress on the tube wall $\tau_w$ versus the nominal shear strain $\gamma_n$ is given in Fig. 5 for selected geometries: $t/d = 0.02$ and 0.15, and $l/d = 0.1, 1$ and 10. The wall stress $\tau_w$ is related to the shear force $F_s$ and the wall cross-sectional area $A = \pi(d - t)t$ according to $\tau_w = F_s/A$. Likewise, $\gamma_n$ has the same definition as that introduced for the solid rod of the previous section, such that $\gamma_n = \psi / l$. The deformed shapes and collapse modes of the selected tubes are catalogued in Table 1. A stable collapse mode of wall shear occurs for thick-walled tubes $t/d = 0.15$ at all three lengths $l/d = 0.1$ to 10: $\tau_w$ increases monotonically with $\gamma_n$ and no plastic instability is detected in the $\tau_w$ versus $\gamma_n$ curves; see Fig. 5b. As $\gamma_n$ approaches the value of unity, orientation hardening dominates the response in the manner revealed by the idealised calculations on the rigid, ideally plastic solid-walled bar, recall Fig. 3.

Now consider the choice $t/d = 0.02$. The collapse mode now varies with the choice of $l/d$: as $l/d$ is increased from 0.1 to 1 and then to 10 the mode switches from wall wrinkling to plastic shear buckling and thence to a plastic bending instability. The buckling modes for each are given in Table 1 at a shear strain of $\gamma_n = 0.2$, and the

<table>
<thead>
<tr>
<th>$l/d$</th>
<th>$\gamma_n$</th>
<th>Deformed shapes</th>
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<tr>
<td>0.1</td>
<td>0.02</td>
<td>Plastic wrinkling (Deformation = 10)</td>
</tr>
<tr>
<td></td>
<td>0.15</td>
<td>Stable shear</td>
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<td>1</td>
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<td>Plastic Shear buckling</td>
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<td>0.15</td>
<td>Plastic bending instability</td>
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<td>Stable shear</td>
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collapse responses are summarized in Fig. 5a. The main features of each of these modes are as follows. Stable shear wrinkling occurs with a monotonically increasing response for \( l/d = 0.1 \). This mode resembles the wrinkling of thin sheet in longitudinal shear, and has been observed experimentally in the longitudinal shear of square honeycomb core by Côté et al. (2006b). In contrast, for \( l/d = 1 \), plastic shear buckling leads to a peak in the collapse response. And at \( l/d = 10 \), a plastic bending instability occurs at the ends of the long, slender tubes; despite the fact that the tube is slender, the high degree of strain hardening in the 304 stainless steel ensures that a graceful collapse response occurs, with only a mild peak in load.

2.3.1. Contour plots of collapse load

The remainder of this section is limited to the development of contour plots of collapse load in the plastic regime, for tubes of practical section \( 0.5 > t/d > 0.01 \). Write \( \tau_{wpk} \) as the maximum value of nominal wall shear stress \( \tau_w \) and write \( \gamma_{pk} \) as the corresponding value of nominal shear strain (as shown in the insert in Fig. 5a).

This definition of \( \tau_{wpk} \) is restricted to the regimes of plastic shear buckling and plastic bending instability for which a load peak occurs. The other two deformation modes (stable wrinkling and stable shear) continuously harden. Fig. 6a shows contours of normalised peak shear strength \( (\tau_{wpk}/\sigma_Y) \) and the corresponding normalised shear strain \( \gamma_{pk} \) on the map with axes \((l/d, t/d)\). The various regimes of behaviour are separated by thick solid lines.

The role of strain hardening is difficult to assess in a precise manner, and this would require the generation of new maps of the type shown in Figs. 4 and 6. However, an indication of the peak wall stress for an elastic, ideally plastic solid is given by (13) with \( \gamma_{pk} = \gamma_n = 0 \).

The structural efficiency of the competing tube designs is assessed in Fig. 6b by plotting contours of the normalised tube strength \( \tau_{wpk}/\sigma_Y \) and mass on the collapse map, as follows. The mass of the tube is given by

\[
m = \frac{\pi}{4} \rho (d^2 - (d - 2t)^2)
\]

(14)

in terms of the density \( \rho \) of wall material. Introduce a reference mass \( m_r \) by considering a solid circular bar of length \( l \) and diameter \( d = l \), such that

\[
m_r = \frac{\pi}{4} \rho l^3
\]

(15)

Then, the mass of the tube can be written in dimensionless form \((m)\) as

\[
m = \frac{m}{m_r} = \left( \frac{d}{l} \right)^2 \left[ 1 - \left( \frac{1 - 2t}{d} \right)^2 \right]
\]

(16)

Note that \( m \) is equal to unity for a solid strut \((2t = d)\) and of diameter \( d = l \). For any fixed value of \( m \) a family of tube geometries \((t/d, l/d)\) exist with equal mass and equal length \( l \), and consequently equal cross-sectional area of wall. The contours of normalised peak shear strength \( \tau_{wpk}/\sigma_Y \) and normalised mass \( m \) are plotted in Fig. 6b. The contours of normalised peak shear strength run almost parallel to those of \( m \) indicating that there is no specific optimum path for the peak wall stress (and thereby for the peak shear force). A similar exercise has been conducted by Pingle et al. (2010) for vertical tubes under axial compression. In that case, a definite optimal path was identified.

3. The pyramidal tube lattice

3.1. Geometry

Recall that a unit cell of hollow pyramidal lattice material, composed of four inclined circular tubes, is shown in Fig. 1a. The geometry is defined in terms of the wall thickness \( t \), outer tube diameter \( d \), tube length \( l \) and inclination \( \omega \) of each strut. The height of the core is \( \sin \omega \). In general, the tube centres are offset by a distance of \( 2k \) as shown in Fig. 1, and tubes touch each at the face-sheets when \( k = k_{min} \). Consequently, \( k \) is constrained such that

\[
k \geq k_{min} = d \sqrt{\frac{1 + \sin^2 \omega}{2 \sin \omega}}
\]

(3.1)

Unless otherwise specified, all results for the pyramidal core discussed subsequently assume that the tubes touch at the apex, \( k = k_{min} \). For arbitrary \( k \), the relative density of the lattice is

\[
\rho = \frac{2 \pi [d^2 - (d - 2t)^2]}{(4k + 2l \cos \omega)^2 \sin \omega}
\]

(3.2)

The direction of relative shearing of the top and bottom faces of the sandwich panel faces is orientated at an angle \( \phi \) to the 1-direction within the 1-2 basal plane, as defined in Fig. 1a. Throughout this

Fig. 6. The collapse mechanism map for a vertical tube in shear. (a) Contours of normalised shear strain at peak load and normalised peak shear strength and (b) contours of normalised shear stress and normalised mass. The boundaries between collapse regimes are re-plotted from Fig. 4.
study, the hollow pyramidal unit cell is sheared in the direction $\phi = 0^\circ$, unless otherwise stated. We shall explore the dependence of the peak shear strength upon $\phi$ in Section 4.

3.2. Finite element modelling of hollow pyramidal core

The finite element simulations were carried out using the implicit time integration version of the FE program ABAQUS/Standard in a similar manner to that used for the vertical tubes. The unit cell of the pyramidal core is sandwiched between two rigid surfaces simulating the face sheets (Fig. 1a). The tube inclination $\omega$ is set to 55°, as employed in the experimental study by Queheillalt and Wadley (2011). In the case of $\phi = 0$, it is sufficient to limit the FE model to two out of the four tubes due to symmetry considerations. For any other value of $\phi$, all four tubes are modelled. Thin tubes ($t/d < 0.03$) are meshed using eight noded, linear hexahedral element (C3D8R in ABAQUS). The average element size equals 10\(\times\)C15 and seven integration points across the thickness are adopted to capture the complex buckling modes. Thick tubes ($t/d > 0.03$) are meshed using eight noded, linear hexahedral element (C3D8R in ABAQUS) with average element size of $t/4$. Mesh sensitivity studies showed that the above choices give a converged solution.

The unit cell is loaded by prescribing displacements $u_1 = u\cos\phi$ and $u_2 = u\sin\phi$ in the 1 and 2 directions respectively, where $u$ is the applied displacement in the direction $\phi$. All degrees of freedom (translational and rotational) on the bottom rigid surface are constrained, and the rotational degrees of freedom of the nodes at the top rigid surface are constrained. The net force in the 3-direction equals zero and the displacement $u_3$ of the top rigid plate is set to 55°. The tube wall material is treated as a J2 flow theory elastic–plastic solid, with material characteristic as plotted in Fig. 2b.

Write the force $S$ as the work conjugate to the applied in-plane displacement $u$ and write $n$ as the number of tubes modeled ($n = 2$ for $\phi = 0$ and $n = 4$ for $\phi \neq 0$). The nominal shear stress $\tau_n$ on the face sheets with a pyramidal core is given by

$$\tau_n = \frac{85}{n(4k + 2l\cos\omega)}$$  \hspace{1cm} (3.3)

while the corresponding nominal strain of the core of the sandwich plate is $\gamma_n = u/(l\sin\omega)$.

No initial geometric imperfections were introduced in the FE models. This is motivated by the fact that the inclined tubes are not loaded along their axes, and the observed buckling modes are imperfection insensitive. (The insensitivity to imperfections was confirmed by performing a limited number of test runs on selected geometries with geometric imperfections in the form of the first eigenmode of elastic buckling.)

4. Performance and collapse mechanism charts for the pyramidal core

For the loading case $\phi = 0$, alternating tubes along the array are compressed or stretched. The compressed tubes collapse in a variety of buckling modes and this dictates the overall shear response of the unit cell. A large number of simulations have been performed (about 150) to explore the dependence of collapse mode upon geometry ($t/d,l/d$), and the observed modes a–f are marked on a collapse mechanism map in Fig. 7a. The regimes of each mode are thereby identified and are replotted in Fig. 7b, along with representative geometries a–f to illustrate the modes A–F. Modes B, C, E and F are the same as those noted previously by Pingle et al. (2010) for the compressive loading of the pyramidal core, and geometries b–f are the same as those considered in the previous study. (The boundaries of the collapse regimes for compressive loading of the pyramidal are included in Fig. 7b). For completeness, the collapse response for the geometries a–f of Fig. 7b is contrasted for shear loading and for compressive loading in Fig. 8. The predicted modes in shear are now catalogued, and the deformed geometries are shown in Table 2.

- **Mode A** is plastic shear buckling. This mode replaces the axisymmetric mode A of Pingle et al. (2010) for the pyramidal core under compressive loading. It is the same mode as the plastic shear buckling of vertical tubes, as plotted in Fig. 4. The collapse response for geometry a is given in Fig. 8a, and this is qualitatively similar to that given in Fig. 5a for $t/d = 0.02$ and $l/d = 1$. We note from Table 2 that all inclined members of the pyramidal core geometry a undergo shear wrinkling, not limited to the compressed inclined tubes. In contrast, the stretched inclined tubes of the pyramidal core behave in a stable manner without buckling for the remaining geometries b–f, see Table 2.
- **Mode B** is stable shear. This is essentially the same mode as the plastic barrelling mode B of Pingle et al. (2010). A hardening response is noted, and the collapse mode is essentially the same as that labelled ‘stable shear’ in Fig. 4 of the current study for the vertical tube under remote shear.

![Fig. 7.](image-url)
It is instructive to add the nominal shear stress versus strain response of the parent 304 stainless steel to Fig. 8a ($\phi = 1$). This allows for an assessment of the knock-down in shear strength of the core due to topology. It is evident that the stable shear mode B has the strongest response, and the various buckling modes give varying degrees of knock-down, with the largest reduction in strength due to elastic buckling, mode F.

4.1. Shear strength of sandwich core as a function of core geometry

Write $\tau_{pk}$ as the peak shear strength of the hollow pyramidal core (i.e. peak value of $\tau_n$). This definition is restricted to modes A, C, E and F where there is a definite peak in the nominal shear stress versus strain response. Fig. 9 contains a design chart for the hollow pyramidal core under shear loading, with axes $l/d$ and $t/d$. It displays contours of peak shear strength $\tau_{pk} / \sigma_y$ within the regime of plastic buckling (modes A, C and E) and contours relative density $\rho$ over the full map. The thick inclined line on the design chart separates the ‘stable shear’ regime, for which no peak strength exists, from the plastic buckling regime that shows a peak strength. The contours of $\rho$ and $\tau_{pk} / \sigma_y$ are almost parallel to each other for all four buckling modes indicating that the shear strength $\tau_{pk} / \sigma_y$ depends primarily upon $\rho$ rather than the two independent parameters $l/d$ and $t/d$. Consequently, there is no optimal path of core geometry that give rise to the peak strength for any given $\rho$. In order to determine the sensitivity of the stable strain-hardening response of the hollow pyramidal core to tube geometry, we plot $\tau_n/(\rho \sigma_y)$ versus $\gamma_n$ in Fig. 10 for selected values of $t/d$ and $l/d = 1$. With this choice of normalisation, the curves almost overlap suggesting that the peak shear strength scales linearly with $\rho$. A regression analysis gives

$$\frac{\tau_{pk}}{\sigma_y} = 0.6 \rho$$

over the regime of plastic buckling. The above behaviour contrasts with that for compression of the hollow pyramidal core: Fig. 12 of Pingle et al. (2010) reveals the existence of an optimal path that maximises the compressive strength for any value of $\rho$.

4.2. Effect of the direction of loading $\phi$ upon collapse response

The above analysis has assumed that the shear direction is aligned with the axis of the unit cell for the pyramidal core, $\phi = 0$. The dependence of $\tau_{pk}$ upon the loading direction $\phi$ is now explored for the selected geometries a, c and e of Table 2, for the type 304 stainless steel. The finite element predictions are reported in Fig. 11, and indicate a small drop in shear strength as $\phi$ is increased from 0 to $\pi/4$. (Each of the curves a, c and e were drawn from 8 finite element simulations.)

The dependence of $\tau_{pk}$ upon $\phi$ has been explored previously by Deshpande and Fleck (2001) for the pyramidal core with solid struts. They considered a rigid-ideally plastic response and obtained the following analytical estimate for the shear strength as a function of the loading angle $\phi$ and angle of inclination of struts $\omega$:

$$\frac{\tau_{pk}}{\rho \sigma_y} = \frac{\sin(2\omega)}{\cos(\pi/4 - \phi)}$$

Since the pyramidal struts are undergoing stretching, the above expression remains unchanged for tubular struts, and is included in Fig. 11. A comparable drop in $\tau_{pk}$ is predicted with increasing $\phi$ to that obtained in the present study. It is also clear from Fig. 11 that strain hardening elevates the peak stress by delaying plastic buckling. We consider (4.2) to be a useful formula for the collapse strength of pyramidal cores made from low strain hardening solids. Its accuracy has been confirmed experimentally by Kooistra and Wadley (2007) for 6061-T6 aluminium alloy.

![Fig. 8. (a) The shear stress versus strain response of six representative inclined tubes a–f. The geometries details and deformed shapes are displayed in Table 2. (b) The compressive response of a sandwich panel containing as core each of the inclined tubes a–f.](image)
5. Discussion of the performance of the hollow pyramidal core

5.1. Comparison of predicted and measured collapse responses

Queheillalt and Wadley (2011) have conducted a limited set of experiments on hollow pyramidal core under shear, in order to measure the collapse response as a function of geometry. The fidelity of the FE simulations is gauged by comparing our predictions with the observed response of two geometries as investigated by Queheillalt and Wadley (2011). Recall that the material response used in the above predictions is based upon those employed by Queheillalt and Wadley (2011). The experimental geometries are labelled as Exp. 1–3 in Fig. 7b. They each possess \( l/d = 4.88 \) but have differing values of \( t/d \). In the experiments, the adopted spacing was \( k = \sqrt{2}d \), as defined in Fig. 1b.

Additional simulations have been performed in order to predict the response for each of the three geometries, and the comparison of nominal shear stress versus shear strain responses, and of deformed geometries, is made in Fig. 12. Excellent agreement is noted between observations and predictions, confirming the fidelity of the FE model for the hollow pyramidal core. Remarkably, the choice of geometries by Queheillalt and Wadley (2011) gave rise to three distinct collapse modes (multi-lobe diamond, mode C; global plastic buckling, mode E; and stable plastic shear). And these three geometries occupy the three requisite domains of the buckling map, as demonstrated in Fig. 7b.

Table 2
Predicted collapse modes of the hollow pyramidal core made from inclined tubes \((\alpha = 55^\circ)\) under shear loading. Deformed geometries of the pyramidal core are shown at a stated value of nominal shear strain \( \gamma_n \). Two out of the four inclined tubes per unit cell are displayed.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Deformed shapes</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td><img src="a" alt="Image" /></td>
<td>Plastic shear buckling</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2} - 0.02 \frac{1}{2} - 0.6 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.040 )</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td><img src="b" alt="Image" /></td>
<td>Stable shear</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2} - 0.4 \frac{1}{2} - 1.0 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.40 )</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td><img src="c" alt="Image" /></td>
<td>Multi-lobe diamond</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2} - 0.02 \frac{1}{2} - 3.0 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.01384 )</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td><img src="d" alt="Image" /></td>
<td>Stable shear</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2} - 0.1 \frac{1}{2} - 3 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.06356 )</td>
<td></td>
</tr>
<tr>
<td>e</td>
<td><img src="e" alt="Image" /></td>
<td>Global plastic buckling</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2} - 0.06 \frac{1}{2} - 14.5 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.000442 )</td>
<td></td>
</tr>
<tr>
<td>f</td>
<td><img src="f" alt="Image" /></td>
<td>Euler buckling</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2} - 0.068 \frac{1}{2} - 94 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \rho = 0.000541 )</td>
<td></td>
</tr>
</tbody>
</table>
5.2. Comparison of competing core topologies

It is instructive to compare the shear strength of the hollow pyramidal core with those of competing topologies. Côté et al. (2006a,b) have performed finite element calculations on the shear response of a range of cores made from type 304 stainless steel, and reported the shear strength at a shear strain of 5%. This criterion is slightly different from the definition of peak shear strength as used in the present study. Thus, in order to make for a fair comparison, the shear strength \( s_Y \) at a shear strain of 5% has been calculated for the hollow pyramidal core of the present study. The comparison is given in Fig. 13:

\[
s_Y = \frac{\tau_{pk}}{\sigma_Y}
\]

\( s_Y \) is plotted as a function of \( /C22q \) for the hollow pyramidal core (\( t/d = 0.05 \)), solid pyramidal core (\( t/d = 0.5 \)), square honeycomb, and for the corrugated core in both longitudinal and transverse shear. For \( /C22q \) exceeding 1% all types of core undergo plastic buckling and have similar shear strengths. But, at lower values of relative density, the prismatic 2D cores (square honeycomb and corrugated cores) undergo elastic buckling whereas the pyramidal core maintains collapse by plastic buckling down to \( /C22q = 0.1 \% \) (not shown in Fig. 13, but deduced from Fig. 9).

5.3. Energy absorption of the hollow pyramidal core

It is clear from Fig. 13 that the hollow pyramidal core has a high shear strength unless the struts are sufficiently slender (\( l/d > 80 \)) for elastic buckling to occur. Further, the plastic buckling response is relatively benign post-peak load, recall Fig. 8a. Consequently, it is anticipated that the hollow pyramidal core is able to absorb a large amount of energy in collapse. This would make it attractive in sandwich panel applications such as vehicle collision and blast mitigation.

Define the shear energy absorption capacity as the work done, per unit volume of the pyramidal core, up to a nominal shear strain of 0.51:

\[
W_s = \int_0^{0.5} \tau_{n} d\gamma_n
\]

The predictions for \( W_s \) are normalized by the factor \( 0.5\rho\sigma_Y \), which is the energy absorbed by an ideally plastic cellular solid of relative density \( \rho \) up to nominal strain of 0.5. This normalized value can thereby be viewed as a structural efficiency for energy absorption. Fig. 14 shows the dependence of normalized energy absorption capacity upon lattice relative density \( /C22q \), for selected values of \( t/d \) in the range 0.05 to 0.5 (the solid section limit). The normalized energy absorption capacity decreases from about unity at \( /C22q = 0.2 \) to a value in the range of 0.01 to 0.08 (depending upon the value of \( t/d \)). The dependence upon \( t/d \) at any given \( \rho \) is non-monotonic: at low \( \rho \) (below 0.01) the highest structural efficiency is obtained at \( t/d = 0.2 \) whereas at high \( \rho \) (above 0.1) the highest structural efficiency is obtained at \( t/d = 0.05 \). The pyramidal core made from solid inclined

---

Footnote: The value of 0.5 is arbitrary, but deemed to be a practical value for the design of an energy-absorber.
struts ($t/d = 0.5$) underperforms compared to the hollow pyramidal cores, particularly at low relative densities.

In broad terms, the energy absorption of the hollow pyramidal core can be quantified as a power-law function of $q/C^2$ for $q$ in the range 0.002 to 0.2 and $t/d$ in the range of 0.05 to 0.2. A best fit to the predictions of Fig. 14 gives:

$$W_s = 9.5 \frac{1}{q/C^2} \sigma_Y$$

(5.2)

We note a stronger dependence of $W_s$ upon $\rho$ than for the shear strength, $\tau_{pk} \propto \rho$, recall (4.1). The stronger dependence upon $\rho$ (ie. the $3/2$ power rather than linear dependence) is due to the fact that stable shearing occurs at high $\rho$ whereas post-buckling softening occurs at low $\rho$.

Open-celled metal foams are competing sandwich core materials, with an attractive energy absorbing capacity due to the fact that they exhibit a strength plateau post yield. The energy absorption capacity of metal foams up to a shear strain of 0.5 is given by (Ashby et al., 2000)

$$W_s = 0.1 \frac{1}{q/C^2} \sigma_Y$$

(5.3)

for open-celled, almost isotropic, metal foams such as Duelcel aluminum foam. This prediction for the energy absorption of metal foams under shear loading has been included in Fig. 14, and can be compared directly with (5.2). It is evident that the functional dependence of absorbed energy upon relative density is the same

for the two topologies, but the level of absorbed energy is significantly higher for the hollow pyramidal core.

6. Concluding remarks

The current study has highlighted the relationship between tube geometry ($l/d$ and $t/d$) and the modes of elastic and plastic buckling for both vertical and inclined AISI 304 stainless steel tubes under shear loading. A similar mapping exercise has been conducted for the compressive response of sandwich cores by Pingle et al. (2010): for the hollow pyramidal core, several of the
collapse modes are the same for loading in compression or shear. In contrast, the collapse mechanisms map for the vertical tube is qualitatively different for shear loading and for axial compression.

The collapse mechanism maps of Figs. 6a and 9 reveal that vertical and inclined tubes undergo stable shear at sufficiently large $d/t$ and sufficiently low slenderness ratio $l/d$. Plastic buckling intervenes and a peak shear strength arises when a transition boundary is crossed to thin-walled and slender tubes. Our study reveals that the peak shear strength for plastic buckling depends primarily upon the relative density of the core rather than the active buckling mode: a change in $l/d$ and $t/d$ at fixed $\rho$ can change the buckling mode but will have only a mild effect upon the peak strength.

The FE predictions agree with the observed buckling modes and collapse responses of the hollow pyramidal core, as measured by Queheillalt and Wadley (2011). It is clear from their previous study, and from the more complete theoretical characterisation of the present study, that the hollow pyramidal core is attractive for sandwich construction. It can be manufactured over a wide range of relative density, and has both high strength and energy absorbing capability.

Acknowledgement

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Appendix A. Analytical formulae for elastic buckling of tubes under shear loading

Tubes of sufficiently thin-wall ($t/d < 0.03$) undergo elastic buckling in shear, and analytical formulae have been developed to quantify the buckling load as a function of geometry ($t/d$ and $l/d$). These formulae have been assembled in the NACA report of Gerard and Becker (1957) and have been used in order to construct the regimes of elastic buckling in Fig. 4; the two elastic buckling modes in shear are an elastic bending instability for slender tubes and elastic shear buckling for stocky tubes. Selected finite element calculations of the elastic bifurcation load, and of the equilibrium path using the finite deformation option within ABAQUS have been performed within the elastic buckling regime in order to verify the accuracy of the analytical expressions. (The same finite element techniques were used as detailed in Section 2.2). The deformed shapes of the tubes in the elastic post-buckled state are included in Fig. 4 and have been generated by the finite element simulations.

(a) Elastic bending instability (Brazier buckling)

Since formulae for elastic Brazier instability are well established (Gerard and Becker, 1957), we use these to estimate the peak loads corresponding to the elastic bending instability. Consider the vertical tube of outer diameter $d$, wall thickness $t$ and length $l$ under transverse shear (Fig. 2a).

The application of a transverse shear force $F_s$ at each end of the tube generates a bending moment which varies linearly along the length of the tube and has the maximum value $M = F_sl/2$ at each end, as demanded by static equilibrium. When the tube is slender and thin-walled, the bending moment $M$ at each end of the tube leads to flattening of the cross-section and thence to a reduction in the effective section modulus of the cylinder. An elastic instability occurs and a peak bending moment is attained. Brazier (1927) calculated the peak moment at which a circular tube becomes unstable due to flattening of the cross-section under pure bending. In the elastic range, the critical bending moment $M_B$ to cause the Brazier instability is

$$M_B = 0.272 \frac{\pi E t^2}{\sqrt{3} \sqrt{1 - \nu^2}}$$  \hspace{1cm} (A.1)

and consequently the shear force for Brazier instability reads

$$F_s = 2M_B \frac{l}{t} = 0.544 \frac{\pi E t^2}{\sqrt{3} \sqrt{1 - \nu^2} l}$$  \hspace{1cm} (A.2)

(b) Elastic shear buckling

Lundquist (1935) conducted tests on the elastic buckling of thin-walled but stocky circular cylinders under transverse shear. He showed that the critical shear force for buckling is

$$F_s = 1.25 \frac{k_t \pi^2 E}{24(1 - \nu^2)} \left( \frac{l}{d} \right)^{1.5} t$$  \hspace{1cm} (A.3)

The factor $k_t$ is the buckling coefficient for cylinders loaded in shear, and as given by Lundquist (1932). He plotted $k_t$ as a function of a geometric factor $L_1$ as defined by

$$Z_l = \frac{l^2}{r^2} \sqrt{1 - \nu^2}$$  \hspace{1cm} (A.4)

and a curve fit to this plot gives

$$k_t \approx 0.88 (Z_l)^{3/4}$$  \hspace{1cm} (A.5)

The boundary between the Brazier instability and shear buckling is obtained by equating the expressions (A.2) and (A.3), to give

$$\left( \frac{l}{d} \right)^{1.5} = 0.0416 \left( \frac{l}{d} \right)^{-1/2}$$  \hspace{1cm} (A.6)

and this boundary has been added to the collapse mechanism map of Fig. 4.

(c) Boundary between elastic and plastic buckling

The boundary between elastic and plastic buckling is obtained by equating the maximum von Mises stress within the tube to the yield strength $\sigma_Y$ of the wall material. Consequently, the boundary between elastic and plastic bending instability is obtained via (A.1) as

$$\frac{t}{\sigma_Y} = 0.329 = 0.0026$$  \hspace{1cm} (A.7)

upon noting that the yield strain $\epsilon_Y$ for annealed AISI 304 stainless steel is $\epsilon_Y = 8.86 \times 10^{-5}$. In similar manner, the boundary between elastic and plastic shear buckling for the stainless steel follows from (A.3) as

$$\frac{t}{\sigma_Y} = 0.00156 \left( \frac{l}{d} \right)^{2/5}$$  \hspace{1cm} (A.8)
References


Further reading