ELASTIC RESPONSE OF A LAYERED CYLINDER SUBJECTED TO DIAMETRAL LOADING

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Abstract—The elastic stress field produced in an arbitrarily layered cylinder subjected to diametrically opposed loads is examined and subsequently employed to investigate the transverse strengths of tungsten-cored boron and carbon-cored silicon carbide fibers measured from diametral compression tests. The displacement formulation of linear elasticity and the local/global stiffness matrix method are used to obtain solutions for stress and displacement components in a composite cylinder consisting of annular linear elastic isotropic shells under the assumption of plane strain. It is found that the presence of the carbon and tungsten cores in the ceramic fibers gives rise to a tensile stress concentration at the core/outer shell interface under diametral compressive loading. This is shown to contribute to a significant loss of apparent transverse strength. The solution also provides insight into first-stage consolidation of alloy-coated ceramic fibers. Elevated hoop stresses are shown to occur at fiber-fiber contacts that result in a reduction in the yield coefficient from that of homogeneous alloy fibers. Thus, the initiation of the yield mechanism of consolidation can be accomplished at lower applied loads when a ceramic fiber is present.

1. INTRODUCTION

Solutions to elastic problems involving both homogeneous and heterogeneous cylinders subjected to diametrically opposed point loads are of technological significance to a number of areas of engineering. For instance, the solution to the homogeneous cylinder problem forms the basis of the so-called Brazilian test used to determine the tensile strength of concrete (Timoshenko, 1934). It has been applied by Krieder and Prewo (1972) to the measurement of the transverse strength of 100–150 μm diameter tungsten-cored boron fibers used in light-weight boron/aluminum composites developed for aerospace applications. Eldridge et al. (1993) have also employed this test method to measure the transverse strength of Textron SCS-type SiC fibers that have a carbon core. These SCS fibers are the basis for a new generation of metal-matrix composites intended for elevated-temperature applications. Attempts to model the consolidation of metal-coated fibers to form composites (Davison and Wadley, 1993) have also encountered a need for a more realistic calculation of the elastic stress field of inhomogeneous cylinders.

Cylindrical bodies subjected to radial loading have received extensive study within the framework of linear elasticity theory. Hertz (1982), Mitchell (1900) and later Hondros (1959) investigated isotropic, homogeneous circular cylinders under diametrically opposed point and distributed loads. Further work in this area was continued by Narodetskii (1947), whose complex variable formulation led to an approximate solution for the elastic field within an isotropic, heterogeneous cylinder subjected to the same loading as that studied by Hertz and Mitchell. However, the precision of this complex variable solution was limited because it was developed using truncated series solutions.
Here, a method for solving plane elasticity problems involving arbitrarily layered cylindrical bodies is developed. The solution technique is based on the local/global stiffness matrix formulation originally developed by Bufler (1971) for analyzing the response of multilayered, isotropic elastic media in rectangular coordinates, and Pindera (1991) to determine the elastic response of composite materials and structures. Most recently, Pindera et al. (1993) showed that it can also be applied to axisymmetric, elastoplastic boundary-value problems and it has been used to investigate the effects of fiber and interfacial-layer morphologies on the evolution of residual stresses in metal-matrix composites. Thus far, it has not been employed to solve plane problems in polar coordinates involving arbitrarily layered cylindrical bodies subjected to non-axisymmetric (i.e., diametral) loading. Herein, the displacement formulation of linear elasticity is utilized to develop expressions for the local stiffness matrix of cylindrical elastic shells under the assumption of plane strain. These matrices, when assembled into the global stiffness matrix for an arbitrarily layered cylinder under diametrically opposed loads, facilitate determination of interfacial displacements and ultimately, distributions of stresses within each shell of the layered cylinder. The approach's utility is demonstrated by applying it to the analysis of the diametral compression of boron-coated tungsten, silicon-carbide-coated carbon and a typical alloy-coated fiber (Ti-6Al-4V coated aluminium oxide).

2. ANALYTICAL FORMULATION

2.1. Development and solution of governing differential equations

The model we consider is the two-dimensional cross-section of an arbitrarily layered composite cylinder, comprised of a solid core and \( N - 1 \) concentric shells, subjected to diametrically opposed distributed loads; see Fig. 1. Each of the individual layers is assumed to be linearly elastic and isotropic. The radius of the central core is designated by \( r_N \) and the outer radius of the outside shell by \( r_l \). For the \( k \)th layer, the inner radius is denoted \( r_{k+1} \) and the outer radius \( r_k \). The composite cylinder is assumed to be sufficiently long for a plane-strain approximation to be applicable.

Within the framework of linear elasticity, this problem's solution is obtained using the displacement formulation. This entails expressing the equilibrium equations in terms

![Fig. 1. Arbitrarily layered composite cylinder subjected to diametrically opposed loads (origin of the polar coordinate system is at the center of the fiber).](image-url)
of displacement gradients using the constituents’ constitutive equations and the strain–displacement relations. The equilibrium equations in polar coordinates are:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) = 0
\]

(1)

\[
\frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r} \sigma_{r\theta} = 0
\]

where \( \sigma_{rr} \), \( \sigma_{\theta\theta} \) and \( \sigma_{r\theta} \) are the radial, hoop and shear stresses, respectively. The plane-strain constitutive equations (i.e. taking \( \varepsilon_{zz} = 0 \)) give:

\[
\sigma_{rr} = \frac{E}{(1 + \nu)(1 - 2\nu)} (v \varepsilon_{\theta\theta} + (1 - \nu) \varepsilon_{rr})
\]

\[
\sigma_{r\theta} = \frac{E}{(1 + \nu)} \varepsilon_{r\theta}
\]

(2)

\[
\sigma_{\theta\theta} = \frac{E}{(1 + \nu)(1 - 2\nu)} (v \varepsilon_{rr} + (1 - \nu) \varepsilon_{\theta\theta})
\]

where \( E \) is the Young’s modulus and \( \nu \) is Poisson’s ratio.

The displacements within each composite layer have the functional form:

\[
u = \nu(r, \theta) \quad v = v(r, \theta)
\]

(3)

Equation (1) can be written in terms of displacements by substituting (2) into (1) and then utilizing the strain–displacement relations:

\[
\varepsilon_{rr} = \frac{\partial u}{\partial r} \quad \varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \varepsilon_{r\theta} = 1 \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{\nu}{r} \right).
\]

(4)

The resulting partial differential equations (Navier’s equations) can be written:

\[
2(1 - \nu) \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) + \frac{(1 - 2\nu)}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial^2 v}{\partial \theta \partial r} + \frac{\nu}{r} \frac{\partial^2 v}{\partial \theta^2} + \frac{(4\nu - 3)}{r^2} \frac{\partial v}{\partial \theta} = 0
\]

\[
(1 - 2\nu) \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) + \frac{2(1 - \nu)}{r^2} \frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} + \frac{3 - 4\nu}{r^2} \frac{\partial u}{\partial \theta} = 0.
\]

(5)

These equations can be solved subject to specified (traction) boundary conditions:

\[
\sigma_{rr}(r_1, \theta) = -\frac{P}{\zeta} \quad \text{for} \quad |\theta - \pi/2| < \frac{\zeta}{2r_1}, \quad |\theta - 3\pi/2| < \frac{\zeta}{2r_1}
\]

\[
\sigma_{rr}(r_1, \theta) = 0 \quad \text{elsewhere}
\]

\[
\sigma_{r\theta}(r_1, \theta) = 0 \quad \text{for all} \ \theta
\]

(6)

and by invoking continuity of both interfacial displacements and tractions:

\[
u_k(r_{k+1}) = u_{k+1}(r_{k+1})
\]

\[
u_k(r_{k-1}) = v_{k+1}(r_{k+1})
\]

\[
(\sigma_{rr})_k(r_{k+1}) = (\sigma_{rr})_{k+1}(r_{k+1})
\]

\[
(\sigma_{r\theta})_k(r_{k+1}) = (\sigma_{r\theta})_{k+1}(r_{k+1})
\]

(7)

where \( P \) is the applied load per unit length and \( \zeta = r_1 \theta_0 \) (see Fig. 1) is a small circumferential distance over which the load is applied, for a fixed angle \( \theta_0 \). It is also required that all the displacements are non-singular within the composite cylinder.
It is convenient to represent the applied radial stress by a Fourier series:

$$\sigma_r(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \quad (8)$$

where $a_0 = -P/r_1 \pi$ and $a_n = -4P(n\pi r_1 \theta_0)^{-1} \sin(n\theta_0/2) \cos(n\pi/2)$.

Solutions to the governing partial differential equations (5) must satisfy the problem's symmetry. Thus, the displacements will be of the form:

$$u(r, \theta) = \sum_{n=0}^{\infty} f_n(r) \cos n\theta \quad v(r, \theta) = \sum_{n=0}^{\infty} g_n(r) \sin n\theta \quad (9)$$

where $f_n(r)$ and $g_n(r)$ are arbitrary radial functions to be determined.

We first consider the special case $n = 0$ (axisymmetric case), for which the coupling between the radial and tangential displacements vanishes. In this case, the radial displacement is only a function of $r$ and the tangential displacement is identically zero. Then, eqn (5) reduces to the single ordinary differential equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{4}{r^2} u = 0. \quad (10)$$

This has the solution:

$$u(r) = A_{10} r + \frac{A_{20}}{r} \quad (11)$$

where $A_{10}$ and $A_{20}$ are arbitrary constants to be determined later.

For $n > 0$, substitution of eqn (9) into eqn (5) yields coupled ordinary differential equations with variable coefficients for the $k$th shell:

$$f_{nk}'(r) + \frac{f_{nk}(r)}{r} - \alpha_{nk} f_{nk}(r) + \frac{n}{2(1 - \nu_k)} \left( g_{nk}(r) \right) \frac{(4\nu_k - 3)}{r^2} g_{nk}(r) - \frac{n}{1 - 2\nu_k} \left( f_{nk}(r) \right) \frac{(3 - 4\nu_k)}{r^2} f_{nk}(r) = 0 \quad (12)$$

where \( \alpha_{nk} = 1 + n^2(1 - 2\nu_k)(2(1 - \nu_k))^{-1} \), $\beta_{nk} = 1 + 2n^2(1 - \nu_k)(1 - 2\nu_k)^{-1}$ and the prime denotes differentiation with respect to $r$. This system of equations may be solved analytically by standard techniques to give:

$$f_{nk}(r) = \sum_{j=1}^{4} \frac{n(\lambda_j + 4\nu_k - 3)}{2\lambda_j (1 - \nu_k)(\alpha_{nk} - \lambda_j^2)} A_{jk} r^{\lambda_j} \quad (13)$$

$$g_{nk}(r) = \sum_{j=1}^{4} \frac{A_{jk} r^{\lambda_j}}{\lambda_j}$$

where $\lambda_1 = n + 1$, $\lambda_2 = n - 1$, $\lambda_3 = -(n + 1)$, $\lambda_4 = -(n - 1)$ and $A_{jk}$ are arbitrary Fourier constants to be determined from the boundary and continuity conditions. We note that the series expressions for the radial functions in eqn (13) are singular at $n = 1$ and thus are not included.

The expressions for the displacements may be simplified further by enforcing symmetry. Thus, $u(r, \theta) = u(r, \pi - \theta)$ and $v(r, \theta) = -v(r, \pi - \theta)$. This implies that the integer $n$ must be even, and so, the final expressions for the displacements in the $k$th shell are:

$$u_k(r, \theta) = A_{10}^k r + \frac{A_{20}^k}{r} + \sum_{n=1}^{\infty} \sum_{j=1}^{4} \frac{n(\lambda_j + 4\nu_k - 3)}{2\lambda_j (1 - \nu_k)(\alpha_{nk} - \lambda_j^2)} A_{jk}^k r^{\lambda_j} \cos n\theta \quad n = 2, 4, \ldots \quad (14)$$

$$v_k(r, \theta) = \sum_{n=1}^{\infty} \sum_{j=1}^{4} \frac{A_{jk}^k r^{\lambda_j}}{\lambda_j} \sin n\theta \quad n = 2, 4, \ldots \quad (15)$$
Equations (14) and (15) are the solution to Navier’s equations (5) for the given loading. Expressions for the radial, hoop and shear stresses in the kth layer are of the form:

\[(\sigma_n)_k = \frac{E_k}{(1 + \nu_k)(1 - 2\nu_k)} \left( A_{10}^k - \frac{(1 - 2\nu_k)}{r^2} A_{20}^k + \sum_{n=2}^{\infty} \sum_{j=1}^{4} R_{jk} A_{jm}^k r^{j-1} \cos n\theta \right) \]  

\[(\sigma_\theta)_k = \frac{E_k}{2(1 + \nu_k)} \sum_{n=2}^{\infty} \sum_{j=1}^{4} S_{jk} A_{jm}^k r^{j-1} \sin n\theta \]  

\[(\sigma_{\theta\theta})_k = \frac{E_k}{(1 + \nu_k)(1 - 2\nu_k)} \left( A_{10}^k + \frac{(1 - 2\nu_k)}{r^2} A_{20}^k + \sum_{n=2}^{\infty} \sum_{j=1}^{4} H_{jk} A_{jm}^k r^{j-1} \cos n\theta \right) \]

where \(n = 2, 4, \ldots\), and \(R_{jk}, S_{jk}\) and \(H_{jk}\) are defined in the Appendix. All that now remains is the evaluation of the Fourier coefficients \(A_{jm}^k\).

2.2. Local/global stiffness matrix formulation

In the local/global stiffness matrix method, the unknown coefficients \((A_{10}^k\) and \((A_{20}^k)\) from the axisymmetric contribution to the stress distribution and the unknown Fourier coefficients, \(A_{jm}^k\), are calculated in terms of the harmonics of the common interfacial displacements between adjacent layers which become the new fundamental unknowns. For the axisymmetric case \((n = 0)\), these coefficients were calculated by Pindera (1991) and subsequently related to the interfacial radial tractions through a local stiffness matrix. For the non-axisymmetric case \((n > 2)\), the Fourier coefficients \(A_{jm}^k\) are expressed in terms of the interfacial displacement harmonics using orthogonality relations:

\[ (u_m^m)^+ = \int_0^{2\pi} u_k(r_k, \theta) \cos m\theta \, d\theta \quad (u_m^m)^- = \int_0^{2\pi} u_k(r_k, \theta) \cos m\theta \, d\theta \]  

\[ (v_m^m)^+ = \int_0^{2\pi} v_k(r_k, \theta) \sin m\theta \, d\theta \quad (v_m^m)^- = \int_0^{2\pi} v_k(r_k, \theta) \sin m\theta \, d\theta \]

where the superscripts \("+"\) and \("-\) refer to the outer and inner surfaces of the layer. Application of the orthogonality relations yields:

\[ (u_m^m)^+ = 2\pi \sum_{j=1}^{4} \frac{m(\lambda_j + 4\nu_k - 3)}{2\lambda_j(1 - \nu_k)(\alpha_{mk} - \lambda_j)} A_{jm}^k \lambda_j \quad m = 2, 4, \ldots \]

\[ (u_m^m)^- = 2\pi \sum_{j=1}^{4} \frac{m(\lambda_j + 4\nu_k - 3)}{2\lambda_j(1 - \nu_k)(\alpha_{mk} - \lambda_j)} A_{jm}^k \lambda_j \quad m = 2, 4, \ldots \]

\[ (v_m^m)^+ = 2\pi \sum_{j=1}^{4} \frac{A_{jm}^k \lambda_j}{\lambda_j} \quad m = 2, 4, \ldots \]

\[ (v_m^m)^- = 2\pi \sum_{j=1}^{4} \frac{A_{jm}^k \lambda_j}{\lambda_j} \quad m = 2, 4, \ldots \]

Expressing these equations in matrix form we have:

\[
\begin{bmatrix}
(u_m^m)^+ \\
(u_m^m)^+
\end{bmatrix} = 2\pi \begin{bmatrix}
\eta_{1k} r_{k1}^\lambda \\
\eta_{2k} r_{k1}^\lambda \\
\eta_{3k} r_{k1}^\lambda \\
\eta_{4k} r_{k1}^\lambda \\
\end{bmatrix} \begin{bmatrix}
\frac{r_{k1}}{\lambda_1} & \frac{r_{k1}}{\lambda_1} & \frac{r_{k1}}{\lambda_1} & \frac{r_{k1}}{\lambda_1} \\
\frac{r_{k2}}{\lambda_2} & \frac{r_{k2}}{\lambda_2} & \frac{r_{k2}}{\lambda_2} & \frac{r_{k2}}{\lambda_2} \\
\frac{r_{k3}}{\lambda_3} & \frac{r_{k3}}{\lambda_3} & \frac{r_{k3}}{\lambda_3} & \frac{r_{k3}}{\lambda_3} \\
\frac{r_{k4}}{\lambda_4} & \frac{r_{k4}}{\lambda_4} & \frac{r_{k4}}{\lambda_4} & \frac{r_{k4}}{\lambda_4} \\
\end{bmatrix} \begin{bmatrix}
A_{1m} \\
A_{2m} \\
A_{3m} \\
A_{4m}
\end{bmatrix}
\]
where \( \eta^m_{jk} = m(\lambda_j + 4\nu_k - 3)(2\lambda_j(1 - \nu_j)(\alpha_{mk} - \lambda_j^2))^{-1} \) for \( j = 1, \ldots, 4 \). For a solid core, the corresponding expressions, given that the displacements at the origin vanish, are:

\[
\begin{pmatrix}
(u^m)^+ \\
(u^m)^-
\end{pmatrix}
= 2\pi
\begin{bmatrix}
\eta^m_{1N} r_1^N & \eta^m_{2N} r_2^N \\
\lambda_1 & \lambda_2
\end{bmatrix}
\begin{pmatrix}
A_{1m} \\
A_{2m}
\end{pmatrix}^N.
\] (26)

Symbolically, eqns (25) and (26) can be written as:

\[
[U^m_k] = [P]_k[A_m]_k.
\] (27)

Similarly, the harmonics of the interfacial stresses can be evaluated from eqns (16) and (17) using the orthogonality relation for the \( m \)th harmonic of the interfacial traction components:

\[
\begin{align*}
(\sigma_{rr})^+_k &= \frac{2\pi}{0} \sigma_{rr}(r_k, \theta) \cos m\theta \, d\theta \\
(\sigma_{\theta\theta})^+_k &= \frac{2\pi}{0} \sigma_{\theta\theta}(r_k, \theta) \sin m\theta \, d\theta \\
(\sigma_{rm})^+_k &= \frac{2\pi}{0} \sigma_{rm}(r_k, \theta) \sin m\theta \, d\theta
\end{align*}
\] (28)

\[
\begin{align*}
(\sigma_{rr})^-_k &= \frac{2\pi}{0} \sigma_{rr}(r_{k+1}, \theta) \cos m\theta \, d\theta \\
(\sigma_{\theta\theta})^-_k &= \frac{2\pi}{0} \sigma_{\theta\theta}(r_{k+1}, \theta) \sin m\theta \, d\theta \\
(\sigma_{rm})^-_k &= \frac{2\pi}{0} \sigma_{rm}(r_{k+1}, \theta) \sin m\theta \, d\theta
\end{align*}
\] (29)

which yields the symbolic relation for the interfacial tractions in terms of the Fourier coefficients

\[
[U^m_k] = [R]_k[\sigma^m_{m}]_k \quad \text{for the } m \text{th harmonic.}
\] (30)

Solving for \( A_m \) in terms of \( U^m_k \) in eqn (27) and substituting the result into eqn (30) yields the local stiffness matrix for the \( k \)th layer, which relates tractions at the top and bottom surface to the corresponding interfacial displacements:

\[
[T^m_k] = [R]_k[\sigma^m_{m}]_k^{-1}[U^m_k] \quad \text{for the } m \text{th harmonic.}
\] (31)

In expanded form, the local stiffness matrix for the \( k \)th layer is given by:

\[
\begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} \\
k_{21} & k_{22} & k_{23} & k_{24} \\
k_{31} & k_{32} & k_{33} & k_{34} \\
k_{41} & k_{42} & k_{43} & k_{44}
\end{bmatrix}
\begin{pmatrix}
(u^m)^+ \\
(u^m)^- \\
(v^m)^+ \\
(v^m)^-
\end{pmatrix}
= \begin{pmatrix}
\sigma_{rr}^- \\
\sigma_{\theta\theta}^- \\
\sigma_{rr}^+ \\
\sigma_{\theta\theta}^+
\end{pmatrix}
\] (32)

For the solid core, the corresponding local stiffness matrix has the form:

\[
\begin{bmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{bmatrix}_N
\begin{pmatrix}
(u^m)^+ \\
(v^m)^+
\end{pmatrix}_N
= \begin{pmatrix}
\sigma_{rr}^- \\
\sigma_{\theta\theta}^+
\end{pmatrix}_N
\] (33)

The elements of the local stiffness matrices for a linearly elastic isotropic (solid) core and an isotropic layer are given in the Appendix. They depend on the elastic and geometric parameters of each region.

By imposing the external boundary conditions, eqn (6), along with continuity of displacements and traction components, eqn (7), a system of equations in the unknown harmonics of the interfacial displacement components arises. The system of equations is formed by first applying eqn (6) and then enforcing eqn (7) at each interface beginning with the outermost shell. This yields:

for the outermost layer:

\[
k_{11}^1 u_1^m + k_{12}^1 v_1^m + k_{13}^1 u_2^m + k_{14}^1 v_2^m = T_1^m
\]

for the \( k \)th interface:

\[
k_{31}^k u_k^m + k_{32}^k v_k^m + (k_{33}^k + k_{11}^{k+1}) u_{k+1}^m + (k_{34}^k + k_{12}^{k+1}) v_{k+1}^m + k_{13}^{k+1} u_{k+2}^m + k_{14}^{k+1} v_{k+2}^m = 0
\] (34)

for the core:

\[
k_{41}^{N-1} u_{N-1}^m + k_{42}^{N-1} v_{N-1}^m + (k_{43}^{N-1} + k_{21}^N) u_N^m + (k_{44}^{N-1} + k_{22}^N) v_N^m = 0
\]
where \( N \) is the number of layers, \((T_m^T)^+ = -4P(mnr, \theta_0)\sin(m\theta_0/2)\cos(m\pi/2)\), and \((u_k^m)^+ = u_k^m\), \((u_k^m)^- = u_k^{m+1}\), etc. Note that in the above and following system of equations, the superscript on the stiffness elements and the subscript on the displacements denotes the layer. The above system of equations can be expressed in terms of the global stiffness matrix for the \( m \)th harmonic:

\[
\begin{bmatrix}
    k_{11} & k_{12} & k_{13} & k_{14} & \cdots & \cdots & \cdots \\
    k_{21} & k_{22} & k_{23} & k_{24} & \cdots & \cdots & \cdots \\
    k_{31} & k_{32} & (k_{33} + k_{11}) & (k_{34} + k_{12}) & \cdots & \cdots & \cdots \\
    k_{41} & k_{42} & (k_{43} + k_{21}) & (k_{44} + k_{22}) & \cdots & \cdots & \cdots \\
    0 & 0 & k_{11} & k_{13} & \cdots & \cdots & \cdots \\
    0 & 0 & k_{21} & k_{23} & \cdots & \cdots & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
    0 & 0 & 0 & 0 & \cdots & (k_{33}^{N-1} + k_{11}^{N}) & (k_{34}^{N-1} + k_{12}^{N}) \\
    0 & 0 & 0 & 0 & \cdots & (k_{43}^{N-1} + k_{21}^{N}) & (k_{44}^{N-1} + k_{22}^{N})
\end{bmatrix}
\begin{bmatrix}
    u_1^m \\
    u_2^m \\
    u_3^m \\
    u_4^m \\
    u_5^m \\
    u_6^m \\
    \vdots \\
    u_N^m \\
\end{bmatrix}
= \begin{bmatrix}
    T_{11}^m \\
    T_{12}^m \\
    \vdots \\
    T_{1N}^m
\end{bmatrix}
\]

Equations (35) are solved for each harmonic of the interfacial displacements which are then substituted into eqn (25) to obtain the unknown Fourier coefficients, allowing the expressions for displacements and stresses within each layer to be calculated.

3. RESULTS AND DISCUSSION

The method developed above can be used to calculate the stresses within diametrically loaded concentric elastic cylinders. We examine a homogeneous cylinder whose distribution of stress has been measured by an optical method. This enables an assessment to be made of the precision of the method. We then examine the stresses developed in "cored" ceramic fibers to better understand their transverse strength testing, and in metal-coated fibers to gain insight into their contact stresses.

3.1. Validation

In order to verify the accuracy of the solution, we first compare the hoop stress for a homogeneous polyester (P-6) fiber calculated by the methods developed here and by Hondros (1959), using a Fourier series representation of Airy's stress function. The results have been generated using 500 terms in the Fourier series expansion for the displacement field given by eqns (14) and (15). This number of terms describes the uniformly-distributed radial tractions applied at the outer radius of the composite cylinder with sufficient accuracy for the employed values of \( \zeta \), while producing convergent displacements and stresses in the interior. Figure 2(a) compares the predicted circumferential stress (normalized by the contact stress \( \sigma_c = P/\zeta \)) as a function of diametral position generated with the present displacement-based approach and Hondros' stress-based solution for \( \zeta = 0.05, 0.10 \) and 0.15. Both methods yield identical results. One also sees that, in the limit as \( \zeta \) tends to 0, the distributed-traction solution tends towards the classical singular point-load solution of Mitchell (1900).

We can also compare the model predictions with the experimental data obtained using the method of optical isodynes (Pindera et al., 1978). Figure 2(b) compares the present solution for three \( \zeta \) values to the Pindera result for the homogeneous polyester fiber analyzed in Fig. 2(a). Best agreement for \( r/r_1 < 0.9 \) is observed when \( \zeta = 0.10 \). It can be seen that near the source of the applied traction \( r/r_1 > 0.9 \), the experimental compressive hoop stress increases at a slower rate than the analytical predictions. One reason may be that in the actual experiment, the fiber was loaded by compressive contact with two other fibers. Thus, the problem was a mixed boundary-value contact
problem with an ill-defined contact distance. Clearly, the present solution closely matches the trends exhibited by the isodyne data, and, in contrast to the solution of Mitchell (1900), does not violate the need for horizontal equilibrium. This suggests that the two-dimensional distribution, obtained here under the assumption of a uniformly distributed radial traction, can be employed to explain results of actual experiments even in regions relatively close to the site of load application.

3.2. Transverse strength analysis

The solution method can be used to analyze the diametral compression testing of ceramic fibers currently used in advanced metal-matrix composites. Experimental data are available for the boron fiber which consists of a tungsten core (actually tungsten boride) onto which a boron shell was deposited via chemical vapor deposition and silicon carbide fibers with a carbon core. The need for analysis is motivated by reports of a marked anisotropy in the axial and transverse strengths of these two fiber types reported by Krieder and Prewo (1972) and Eldridge et al. (1993). Krieder and Prewo used a diametral compression test to experimentally determine the transverse strength of 101 and 145 μm
(4 and 5.7 mil) diameter boron fibers with a core diameter of 12.7 \mu m. They found that the transverse tensile strength of the 101 \mu m fibers was often surprisingly low (207 MPa) compared to the axial strength (3102 MPa), indicating a severe anisotropy in strength. A lesser, but still significant, anisotropy was found for the 145 \mu m diameter fiber.

In the Krieder-Prewo test method, the fiber was loaded in diametral compression and the equatorial tensile hoop stress caused the fiber to split perpendicular to the loading direction. The transverse strength was calculated from the load to cause splitting using the classical solution of Mitchell (1900) for a homogeneous fiber. This predicts a uniform distribution of the circumferential stress along the diametral plane for a homogeneous cylinder, and neglects the potential influence of the tungsten core on the diametral hoop-stress distribution. The low transverse strength was attributed entirely to residual stresses induced during the fabrication process and to the presence of a population of flaws at the core/shell interface. Using a similar analysis Eldridge and co-workers (1993) have measured an even larger anisotropy (3790 MPa for axial loading vs 135 MPa for transverse loads) for Textron's SCS-6 silicon carbide fibers.

We can analyze both the boron and silicon carbide fibers as two-layered composite cylinders. We ignore the microstructural complexity of the outer layers reported by Ning and Pirouz (1991) and Wawner (1988) since data for the different microstructures are unavailable. The geometry and material properties of the cores and shells of both fibers are given in Table 1. Note that the mismatch in material properties between the core and the outer ring for the boron fiber is significantly smaller than that of the silicon carbide fiber.

The global stiffness matrix for a two-layered composite cylinder used to generate the circumferential stresses along the diametral plane of the composite fibers is:

\[
\begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} \\
k_{21} & k_{22} & k_{23} & k_{24} \\
k_{31} & k_{32} & k_{33} & k_{34} \\
k_{41} & k_{42} & k_{43} & k_{44} \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\end{bmatrix}
= \begin{bmatrix}
(T_1^m) \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

where the superscript on the stiffness elements and the subscript on the displacements denote the layer.

Figure 3 shows the hoop stress (concentration) along the diametral line for the boron fibers due to diametral compressive loading. The presence of the more compliant tungsten core results in a tensile stress concentration of approximately 1.35 at the tungsten core–boron shell interface which will lead to an "apparent" loss of transverse strength, especially in ceramic materials where the Weibull modulus can be large. The average hoop stress along the diametral plane of the smaller-diameter fiber is slightly greater than for the larger fiber, although the differences are not very great. However, the percentage of the total cross-sectional area along the diametral plane exposed to a higher circumferential stress is significantly greater in the smaller fiber.

<table>
<thead>
<tr>
<th>Material</th>
<th>Young's modulus (GPa)</th>
<th>Poisson's ratio</th>
<th>Coefficient of thermal expansion ((^\circ C)^{-1})</th>
<th>Normalized layer radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al\textsubscript{2}O\textsubscript{3}</td>
<td>390.0</td>
<td>0.24</td>
<td>-</td>
<td>0.0-0.7</td>
</tr>
<tr>
<td>Ti6Al4V</td>
<td>110.0</td>
<td>0.324</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Boron shell</td>
<td>413.0</td>
<td>0.20</td>
<td>8.5 \times 10^{-6}</td>
<td>1.0</td>
</tr>
<tr>
<td>Tungsten core</td>
<td>379.0</td>
<td>0.28</td>
<td>4.6 \times 10^{-6}</td>
<td>0.125 small</td>
</tr>
<tr>
<td>SiC shell (case I)</td>
<td>413.6</td>
<td>0.17</td>
<td>see Pindera et al. (1993)</td>
<td>1.0</td>
</tr>
<tr>
<td>SiC shell (case II)</td>
<td>551.5</td>
<td>0.17</td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>SiC shell (case III)</td>
<td>413.6</td>
<td>0.17</td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>Carbon core (case I)</td>
<td>34.5</td>
<td>0.24</td>
<td></td>
<td>0.232</td>
</tr>
<tr>
<td>Carbon core (case II)</td>
<td>34.5</td>
<td>0.24</td>
<td></td>
<td>0.232</td>
</tr>
<tr>
<td>Carbon core (case III)</td>
<td>0.689</td>
<td>0.24</td>
<td></td>
<td>0.232</td>
</tr>
</tbody>
</table>
In addition to the stress concentration produced by the applied load, residual stresses induced during fabrication are important and must be included as well, as discussed by Krieder and Prewo. Using the residual stress model of Pindera and Freed (1992), the stress concentration in the circumferential stress at the core/outer sheath interface produced by a temperature change is found to be substantial (Fig. 4) because of the large mismatch in the thermal expansion coefficients of the tungsten core and the boron sheath (see Table I). As in the case of transverse loading, the boron sheath of the smaller-diameter is exposed to a higher circumferential stress than the boron sheath in the larger-diameter fiber.

A second "growth" source of residual stress has been found in boron fibers. It arises during the deposition process because of atomic rearrangements in the deposited structure (Wawner, 1988). The thermal and growth-process-induced residual stresses further increase the tensile hoop stress concentration at the core/outer sheath interface. Thus a large "apparent" difference in the transverse and longitudinal stresses at failure is to be expected if calculations are based on a homogeneous cylinder's elastic field.

Fig. 3. Hoop-stress distribution along the diametral plane of a tungsten-cored boron fiber subjected to diametral compression. Note the small tensile stress concentration at the tungsten core/boron shell interface.

Fig. 4. Thermal component to the residual stress along the diametral plane of a boron fiber subjected to a temperature change.
Figure 5 shows the hoop-stress distribution (concentration) along the diametral line of the silicon carbide fiber subjected to diametral compression. In view of the uncertainty in the material properties of the carbon core and the silicon carbide shell, we give stress profiles for three combinations of material properties for the two regions. Cases I and II approximate the properties of the two regions based on currently available data (Pindera et al., 1993), and differ only in the Young's modulus for the silicon carbide shell. In case III, a very small value of Young's modulus is assigned to the core to simulate a hollow cylinder under diametrically opposed point loads for which a solution has been obtained by Ripperger and Davis (1947).

In contrast with the stress distributions calculated for the boron fibers, the stress concentrations at the carbon core/silicon carbide shell interface are substantially higher. For cases I and II, the stress concentration factor is 6.297 and 6.695, respectively, and 8.146 for case III. By comparison, the magnitude of the stress concentration factor obtained from the solution of a hollow cylinder is 8.199, which agrees well with the result of case III.

The large stress concentrations are a direct consequence of the large mismatch in the Young's moduli of the carbon core and the silicon carbide shell. The greater this modulus mismatch, the greater the stress concentration becomes, with the largest stress concentration occurring when the Young's modulus of the core is zero. The above results clearly indicate that calculation of the transverse tensile strength of the silicon carbide fiber based on the assumption of a uniform circumferential stress along the diametral plane will produce seriously erroneous results. For example, the transverse load necessary to split an as-received SCS6 silicon carbide fiber in diametral compression produces an average hoop stress along the diametral line of 135 MPa if the stress concentration is ignored. Incorporating the stress concentration factor of 6.297 found here into the transverse strength calculation, Eldridge et al. (1993) obtained a value of 850 MPa, substantially reducing the discrepancy between axial and transverse strengths. In the presence of a population of radial microcracks or flaws, such point-wise strength analysis will be modified, possibly further reducing the discrepancy.

In addition to the effect of a non-uniformly distributed hoop stress due to the applied mechanical load, there may also exist fabrication-induced residual stresses like those of the tungsten/boron fiber. However, for this case the situation is not as clear due to a lack of accurate thermal expansion data for the silicon carbide region and the presence of deposition stresses. In particular, the morphology of the silicon carbide fiber plays an important role in the evolution of residual cool-down stresses. For example, a circumferentially orthotropic pyrolitic graphite coating between the carbon core and the
silicon carbide sheath (with a high thermal expansion coefficient and low Young's modulus in the radial direction) acts as a compensating layer and must be included in calculating the residual stress at the carbon core–silicon carbide shell interface. As demonstrated by Pindera and Freed (1992), the distribution of the residual circumferential stress after cool-down depends critically on the value of the thermal expansion coefficient of the silicon carbide regions. For isotropic thermal expansion behavior, Pindera and Freed show that the thermal contribution to the residual stress is very small and uniformly distributed. When the thermal expansion coefficient in the radial direction is 25% greater than that in the circumferential direction, the hoop stress is tensile in the silicon carbide shell and increases near the interface to a maximum of approximately 300 MPa. For circumferentially orthotropic thermal expansion behavior characterized by a thermal expansion coefficient in the radial direction that is 25% smaller than that in the circumferential direction, the opposite trend is observed, and the hoop stress is compressive with a magnitude of about 300 MPa at the interface. In the first and second instances therefore, the differences between the apparent longitudinal and transverse strengths would be substantial, whereas in the third, the high stress concentration at the carbon/silicon carbide interface due to the mechanically applied load would be somewhat offset by the thermally induced compressive hoop stress.

3.3. Consolidation of alloy-coated fibers

There is growing interest in fabricating fiber-reinforced composites by a two-step process. First, (SiC or Al₂O₃) ceramic fibers are uniformly coated with an alloy by a physical vapor deposition process. Then arrays of these coated fibers are compressed to cause yielding and creep deformation of the alloy at fiber–fiber contacts and thus composite densification. Yielding begins at an applied load corresponding to the point at which the contact stress reaches the yield condition for the system (Gampala et al., 1993). Initially, when the applied loads are small, the deformation of both the alloy shell and the ceramic fiber will be purely elastic. The analysis developed above can then be used to identify the effect of the fiber in perturbing the contact stress state of homogeneous alloy fibers, and provides insight into the dependence of the contact yield criterion on ceramic fiber fraction.

Using the material parameters given in Table 1, Fig. 6 shows the calculated hoop-stress distribution along the diametral plane of a representative system consisting of Al₂O₃ fibers coated with Ti-6Al-4V alloy to form composite fibers with either 25% or 49% fiber-volume fraction. Also shown is the result for a homogeneous alloy fiber. The hoop

Fig. 6. Hoop stress profile along the diametral plane for Al₂O₃ fibers coated with a Ti-6Al-4V matrix shell.
stress at the fiber/alloy interface is compressive because the elastic stiffness of the core (i.e. Al₂O₃ fiber) is greater than that of the outer shell (Ti-6Al-4V matrix).

A rise in the hoop stress is seen to occur near the contact when a ceramic fiber is introduced. This raises the Mises stress and causes the onset of plastic yielding to occur earlier than if there were no reinforcement present (Davison and Wadley, 1993). For the 25% fiber-volume fraction case, a factor of 1.47 rise in stress (over the homogeneous fiber) is seen, while for the 49% fiber-volume fraction case, the factor is 1.54. This rise in hoop stress facilitates subsequent plastic deformation of the alloy at proportionally smaller consolidation pressures. Thus, the presence of the ceramic fiber significantly reduces the applied loads that will be required to initiate yielding and the onset of densification.

It is of interest to examine results for the normalized radial stress for \( \theta = 45^\circ \), Fig. 7. The presence of the fiber increases the radial stress in the matrix phase relative to the radial stress in a homogeneous cylinder. In the case where the fiber is 49% of the total volume fraction, a tensile radial stress is developed at the fiber/matrix interface. This tensile stress creates the possibility of debonding between the two regions, potentially resulting in "earring" defects near the equator of the ceramic fiber if the interface is weak.

4. CONCLUSIONS

An efficient method has been outlined for the determination of the elastic response of arbitrarily layered composite cylinders subjected to diametrically opposed loads. This method extends a previously employed local/global stiffness matrix formulation to non-axisymmetric polar problems by the use of a Fourier series expansion technique. Analytical expressions have been given for the displacements and local stiffness matrices of isotropic shells. The method's results correlate well with other calculations and experimental isodyne data for the hoop-stress distribution along the diametral plane of a homogeneous fiber.

The solution method has been applied to the analysis of the transverse strength of boron and silicon carbide fibers. The transverse strength is shown to be influenced by a concentration in the hoop stress at the core/shell interface. The method has also been used to explore the elastic contact of alloy-coated ceramic fibers. A significant elevation of the hoop stress at the contact is found; a result that indicates yielding will occur at lower applied loads than would be needed for homogeneous metal fibers.

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REFERENCES


APPENDIX

The elements of the local stiffness matrix for an isotropic core are given by:

\[ k_{11} = \frac{E_N}{1 - \nu_k} \left[R_{12} r_k^{\nu_1 - 1} X_{11} + R_{22} r_k^{\nu_1 - 1} X_{12}\right] \]

\[ k_{12} = \frac{E_N}{1 - \nu_k} \left[R_{12} r_k^{\nu_1 - 1} X_{12} + R_{22} r_k^{\nu_1 - 1} X_{11}\right] \]

\[ k_{22} = \frac{E_N}{1 - \nu_k} \left[S_{12} r_k^{\nu_1 - 1} X_{11} + S_{22} r_k^{\nu_1 - 1} X_{12}\right] \]

\[ k_{22} = \frac{E_N}{1 - \nu_k} \left[S_{12} r_k^{\nu_1 - 1} X_{12} + S_{22} r_k^{\nu_1 - 1} X_{11}\right] \]

The elements of the stiffness matrix for an isotropic shell are given by:

\[ k_{11} = \frac{E_k}{1 - \nu_k} \left[R_{11} r_k^{\nu_1 - 1} D_{11} + R_{21} r_k^{\nu_1 - 1} D_{12} + R_{11} r_k^{\nu_1 - 1} D_{21} + R_{11} r_k^{\nu_1 - 1} D_{31}\right] \]

\[ k_{12} = \frac{E_k}{2(1 - \nu_k)} \left[S_{11} r_k^{\nu_1 - 1} D_{11} + S_{21} r_k^{\nu_1 - 1} D_{12} + S_{11} r_k^{\nu_1 - 1} D_{21} + S_{11} r_k^{\nu_1 - 1} D_{31}\right] \]

\[ k_{22} = \frac{E_k}{1 - \nu_k} \left[R_{11} r_k^{\nu_1 - 1} D_{12} + R_{21} r_k^{\nu_1 - 1} D_{12} + R_{21} r_k^{\nu_1 - 1} D_{21} + R_{21} r_k^{\nu_1 - 1} D_{31}\right] \]

\[ k_{22} = \frac{E_k}{2(1 - \nu_k)} \left[S_{11} r_k^{\nu_1 - 1} D_{12} + S_{21} r_k^{\nu_1 - 1} D_{12} + S_{11} r_k^{\nu_1 - 1} D_{21} + S_{11} r_k^{\nu_1 - 1} D_{31}\right] \]

where \( R_{12} = \eta_{12} (\nu_k + (1 - \nu_k) \lambda_j) \) \((\nu_k m / \lambda_j)\), \( H_{12} = \eta_{12} (\nu_k \lambda_j + (1 - \nu_k)) + ((1 - \nu_k) m / \lambda_j)\), and \( S_{12} = 1 - \eta_{12} \). The coefficients \( X_j \) are defined by:

\[ X_{11} = P_{22} / \Phi \]

\[ X_{12} = -P_{12} / \Phi \]

\[ X_{21} = -P_{21} / \Phi \]

\[ X_{22} = P_{11} / \Phi \]
where $P_{mn}$ are the elements in eqn (26) and $\Phi$ is the determinant of the matrix. The coefficients $D_{jk}$ are defined below as:

$$D_{jk} = a_{jk} / \Delta$$

where $\Delta$ is the determinant of the matrix in eqn (25) and $a_{jk}$ are the cofactors of the matrix which are defined as:

$$a_{11} = f_{22}(f_{33}f_{44} - f_{34}f_{43}) - f_{23}(f_{32}f_{44} - f_{34}f_{42}) + f_{24}(f_{32}f_{43} - f_{33}f_{42})$$
$$a_{12} = -f_{21}(f_{34}f_{43} - f_{33}f_{44}) + f_{23}(f_{31}f_{44} - f_{34}f_{41}) - f_{24}(f_{31}f_{43} - f_{33}f_{41})$$
$$a_{13} = f_{23}(f_{32}f_{44} - f_{34}f_{42}) - f_{22}(f_{31}f_{44} - f_{34}f_{41}) + f_{24}(f_{31}f_{42} - f_{32}f_{41})$$
$$a_{14} = -f_{23}(f_{32}f_{43} - f_{33}f_{42}) + f_{22}(f_{31}f_{43} - f_{33}f_{41}) - f_{24}(f_{31}f_{42} - f_{32}f_{41})$$
$$a_{21} = -f_{12}(f_{34}f_{43} - f_{33}f_{44}) + f_{13}(f_{32}f_{44} - f_{34}f_{42}) - f_{14}(f_{32}f_{43} - f_{33}f_{42})$$
$$a_{22} = f_{11}(f_{34}f_{43} - f_{33}f_{44}) - f_{13}(f_{31}f_{44} - f_{34}f_{41}) + f_{14}(f_{31}f_{43} - f_{33}f_{41})$$
$$a_{23} = -f_{11}(f_{32}f_{44} - f_{34}f_{42}) + f_{12}(f_{31}f_{44} - f_{34}f_{41}) - f_{14}(f_{31}f_{42} - f_{32}f_{41})$$
$$a_{24} = f_{11}(f_{32}f_{43} - f_{33}f_{42}) - f_{13}(f_{31}f_{43} - f_{33}f_{41}) + f_{14}(f_{31}f_{42} - f_{32}f_{41})$$
$$a_{31} = f_{12}(f_{34}f_{43} - f_{33}f_{44}) - f_{13}(f_{32}f_{44} - f_{34}f_{42}) + f_{14}(f_{32}f_{43} - f_{33}f_{42})$$
$$a_{32} = -f_{11}(f_{32}f_{44} - f_{34}f_{42}) + f_{13}(f_{31}f_{44} - f_{34}f_{41}) - f_{14}(f_{31}f_{42} - f_{32}f_{41})$$
$$a_{33} = f_{11}(f_{32}f_{43} - f_{33}f_{42}) - f_{12}(f_{31}f_{43} - f_{33}f_{41}) + f_{14}(f_{31}f_{42} - f_{32}f_{41})$$
$$a_{34} = -f_{11}(f_{32}f_{44} - f_{34}f_{42}) + f_{12}(f_{31}f_{44} - f_{34}f_{41}) - f_{13}(f_{31}f_{42} - f_{32}f_{41})$$
$$a_{41} = -f_{12}(f_{34}f_{43} - f_{33}f_{44}) + f_{13}(f_{32}f_{44} - f_{34}f_{42}) - f_{14}(f_{32}f_{43} - f_{33}f_{42})$$
$$a_{42} = f_{11}(f_{34}f_{43} - f_{33}f_{44}) - f_{13}(f_{32}f_{44} - f_{34}f_{42}) + f_{14}(f_{32}f_{43} - f_{33}f_{42})$$
$$a_{43} = -f_{11}(f_{34}f_{42} - f_{32}f_{44}) + f_{13}(f_{32}f_{44} - f_{34}f_{42}) - f_{14}(f_{32}f_{43} - f_{34}f_{42})$$
$$a_{44} = f_{11}(f_{34}f_{42} - f_{32}f_{44}) - f_{13}(f_{32}f_{44} - f_{34}f_{42}) + f_{14}(f_{32}f_{43} - f_{34}f_{42})$$

where $f_{mn}$ are the respective terms in the matrix in eqn (25).